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NEARLY CIRCULAR SHEAR MODE CRACKS

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Abstract—In this paper we study the elasticity problems of a planar crack, lying in an infinite three-dimensional solid, the front of which differs only slightly from a circle. The crack system is subjected to loadings that induce shear mode stress intensity factors at the crack front. Quantities such as relative crack surface displacement, in-plane shear mode intensity factor K_2 and anti-plane shear mode intensity factor K_3 are derived in detail. The method used is based on a perturbation technique developed by Rice (*J. Appl. Mech.* **52,** 571–579 (1985)) of calculating the first-order variation of the elastic field of a crack when its front is perturbed from some regular reference geometry. The configurational stability problems of harmonic wave form perturbations of the front of a circular crack under axisymmetric shear loading are studied using the derived formulae. The shape that a planar crack under remote uniform shear loading would take so that the energy release rate distributes uniformly along the crack front is discussed by calculating proper perturbations on a circular crack that meets the above requirement.

INTRODUCTION

For a circular planar crack in an infinite three-dimensional solid, solutions for the stress intensity factors induced along the crack front by various load systems exist in the literature (Tada *et al.*, 1973; Kassir and Sih, 1975; Bueckner, 1977, 1987). Specifically, the solutions for the intensity factor distribution along the circular crack front induced by point force pairs at an arbitrary location on the crack faces, which corresponds to the three-dimensional crack face weight functions of Bueckner (1972) and Rice (1972), generalizing Bueckner's (1970) two-dimensional concepts, are of interest. These solutions were completely derived by Bueckner (1987) for arbitrary point force pairs acting on the crack faces that induce general mixed mode stressing along the crack front, although the solution for a "wedging" force pair that induces mode 1 tension along the crack front was presented earlier by several authors (Tada *et al.*, 1973; Cherepanov, 1979; Bueckner, 1977). Hence by integration of the crack face weight functions we are able to calculate the intensity factors under any load systems for a perfectly circular crack. These solutions, in the limit when the radius of the circular crack approaches infinity, should reduce to the corresponding formulae for a half-plane crack. Solutions for a half-plane crack have been derived by many authors and can be found in Tada *et al.* (1973).

Rice (1985a) developed a method of using the crack face weight function solutions to solve for the elastic field of a crack with a front close to some reference geometry, to first-order accuracy in the deviation of the actual crack from that reference shape. Using that method one can carry out the calculations of the variation of various quantities such as relative crack surface displacements and stress intensity factor distributions when the crack front is perturbed from the reference front to the actual front, if the crack face weight functions for a crack of the reference shape are known in advance.

Rice (1985a) also studied a half-plane tensile crack with a near straight front. In that paper he derived in detail the formulae for the variation in crack opening displacement and stress intensity factors to first-order accuracy in the deviation of the actual crack front from a reference straight line. The shear mode intensity factors for a half-plane crack with a slightly curved crack front were derived by Gao and Rice (1986) using the perturbation method. Gao and Rice (1987) further studied the elasticity problems of somewhat circular tensile cracks. In that paper a full solution, accurate to first order in the deviation of the actual crack front from a circle, is derived for the stress intensity factor distributions and the crack opening displacement. One could verify that those perturbation formulae, in the limit when the radius of the reference crack approaches infinity, also reduce to the corresponding results given by Rice (1985a) for a half-plane crack. Comparison between the exact solutions for an elliptical crack and the numerical results computed from the perturbation formulae when viewing the elliptical crack as perturbed from a circular crack suggested that the perturbation analysis, while theoretically exact only to first order, can be used to produce acceptable results for some planar cracks the shapes of which deviate appreciably from a reference geometry.

Using the mode 2 and mode 3 crack face weight functions derived by Bueckner (1987), we extend in this paper the perturbation analysis to the shear mode circular cracks. Quantities such as in-plane (mode 2) and anti-plane (mode 3) shear stress intensity factors, K_2 and K_3, along a slightly non-circular crack front are solved to similar first-order accuracy.

RELATIVE CRACK SURFACE DISPLACEMENT

Consider a crack lying in an infinite three-dimensional elastic solid with a bounding curve c. The elastic solid is assumed to be homogeneous, isotropic, symmetric about the crack plane and subjected to a general loading system consisting of some distribution of fixed forces that induce mixed mode stress intensity factors along the crack front. A Cartesian coordinate system x, y, z is attached so that the crack plane lies on $y = 0$. In this circumstance Rice (1985a) showed that the variation in the relative displacement $\Delta u_j(x, z)$ $(j = x, y, z)$ between upper and lower crack faces at location $x, 0^{\pm}, z$ when the crack front is altered by $\delta a(s)$, along the normal direction at an arc length location s along the crack front, in the presence of the fixed load system is

$$\begin{aligned}\delta[\Delta u_j(x,z)] &= \oint_c \left\{2\frac{(1-\nu^2)}{E}[K_1^0(s)k_{1j}(s;x,z)+K_2^0(s)k_{2j}(s;x,z)]\right.\\ &\quad \left. +2\frac{(1+\nu)}{E}K_3^0(s)k_{3j}(s;x,z)\right\}\delta a(s)\,\mathrm{d}s \qquad (1)\\ &= 2\frac{(1-\nu^2)}{E}\oint_c\left(\sum_{\alpha=1}^{3}A_\alpha K_\alpha^0(s)k_{\alpha j}(s;x,z)\right)\delta a(s)\,\mathrm{d}s\end{aligned}$$

to first order in $\delta a(s)$. Here $K_\alpha^0(s)$ $(\alpha = 1, 2, 3)$ is the mode α intensity factor distribution induced along the reference crack front by the fixed load system and the crack face weight function $k_{\alpha j}(s; x, z)$ is defined as the mode α intensity factor that would be induced at position s along the reference crack front by a unit force pair in the $\pm j$-direction acting at location $x, 0^+, z$ and $x, 0^-, z$. The constant coefficients $A_1 = A_2 = 1$, $A_3 = 1/(1-\nu)$ are introduced for conciseness in writing formulae. The intensity factor K_α is defined so that $K_\alpha/\sqrt{(2\pi\varepsilon)}$ is the asymptotic form of the relevant singular stress for mode α at a small distance ε ahead of the crack tip on the prolongation of the crack plane. We also define in general that shear mode stress intensity factors have the same signs as those of relative crack surface displacements very near the crack front. In this situation, the following asymptotic formulae are valid:

$$\Delta u_n \sim \frac{8(1-\nu^2)}{E}\sqrt{\left(\frac{\rho}{2\pi}\right)}K_2; \quad \Delta u_t \sim \frac{8(1+\nu)}{E}\sqrt{\left(\frac{\rho}{2\pi}\right)}K_3 \qquad (2)$$

where n and t are the normal and tangential directions along the actual crack front with n lying in the crack plane and ρ is the distance as measured from the crack front in the negative normal direction. Equation (2) is understood to be a general asymptotic relation very near the crack front and will be used in a later section to extract the stress intensity factors from the near tip behavior of the relative crack surface displacement.

To study the shear mode solutions for a nearly circular crack, we shall conveniently choose a circle of radius a to be the reference front of the planar crack and adopt the polar coordinates in the x, z plane with the origin of the polar coordinates located at the center of the reference circular crack. Note that in this coordinate system $s \to a\theta'$ in eqn (1). Therefore, quantities such as $K_\alpha^0(s)$, $k_{\alpha j}(s; x, z)$ and $\Delta u_j(x, z)$ $(j = x, y, z)$ are then replaced by $K_\alpha^0(\theta')$, $k_{\alpha j}(\theta'; r, \theta)$ and $\Delta u_j(r, \theta)$ $(j = r, \theta, y)$, respectively. Let us further introduce the following notation after previous work (Gao and Rice, 1987) when referring to a perfectly circular crack with radius a

$$K_\alpha^0(\theta') = K_\alpha^0[\theta';a]; \quad k_{\alpha j}(\theta';r,\theta) = k_{\alpha j}(\theta';r,\theta;a); \quad \Delta u_j(r,\theta) = \Delta u_j^0[r,\theta;\alpha] \tag{3}$$

where the dependence on the radius of the circular crack is explicitly emphasized. Equation (1) therefore becomes

$$\delta[\Delta u_j(r,\theta)] = 2\frac{(1-\nu^2)}{E}\int_0^{2\pi}\left\{\sum_{\alpha=1}^{3} A_\alpha K_\alpha^0[\theta';a]k_{\alpha j}(\theta';r,\theta;a)\right\}\delta a(\theta')a\ \mathrm{d}\theta'. \tag{4}$$

Equation (4) represents the change in relative displacement $\Delta u_j(r,\theta)$ when the crack is perturbed from the reference circular front of radius a to the actual crack front by an amount equal to $\delta a(\theta')$ at location θ' along the crack front.

If we allow for a uniform perturbation of the circular crack, i.e. $\delta a(\theta') = \delta a$ and divide both sides of eqn (4) by δa, then by letting $\delta a \to 0$ we obtain

$$\frac{\partial \Delta u_j^0[r,\theta;a]}{\partial a} = 2\frac{(1-\nu^2)}{E}\int_0^{2\pi}\left\{\sum_{\alpha=1}^{3} A_\alpha K_\alpha^0[\theta';a]k_{\alpha j}(\theta';r,\theta;a)\right\}a\ \mathrm{d}\theta'. \tag{5}$$

Integrate eqn (5) with respect to crack size variable a', and note $\Delta u_j^0[r,\theta;a'] = 0$ for $r \geqslant a'$ since displacements should be continuous at the crack edge. Hence

$$\Delta u_j^0[r,\theta;a] = 2\frac{(1-\nu^2)}{E}\int_r^a\int_0^{2\pi}\left\{\sum_{\alpha=1}^{3} A_\alpha K_\alpha^0[\theta';a']k_{\alpha j}(\theta';r,\theta;a')\right\}a'\ \mathrm{d}\theta'\ \mathrm{d}a'. \tag{6}$$

By the law of superposition we also have

$$K_\alpha^0[\theta';a] = \int_0^a\int_0^{2\pi}\sum_{i=1}^{3} p_i(\rho,\phi)k_{\alpha i}(\theta';\rho,\phi;a)\rho\ \mathrm{d}\rho\ \mathrm{d}\phi \tag{7}$$

where $p_i(\rho,\phi)$ represent the loads in the j-direction on the crack faces. Equation (7) remains valid for the general loading system if $p_i(\rho,\phi)$ are equal to the stresses that would be induced at the crack site in the absence of a crack. Substituting eqn (7) into eqn (6) and switching the order of integration over a', θ' with ρ, ϕ lead to

$$\Delta u_j^0[r,\theta;a] = \int_0^{2\pi}\int_0^a\sum_{i=1}^{3} p_i(\rho,\phi)D_{ji}(r,\theta;\rho,\phi)\rho\ \mathrm{d}\rho\ \mathrm{d}\phi \tag{8}$$

where

$$D_{ji}(r,\theta;\rho,\phi) = \int_{\max(r,\rho)}^a\int_0^{2\pi}\left(\sum_{\alpha=1}^{3} A_\alpha k_{\alpha i}(\theta';\rho,\phi;a')k_{\alpha j}(\theta';r,\theta;a')\right)a'\ \mathrm{d}a'\ \mathrm{d}\theta'. \tag{9}$$

The function $D_{ji}(r,\theta;\rho,\phi)$ is identified as the general crack face Green's function for a circular crack. By definition it is the relative displacement induced at location r, $0^{\pm}$, θ on the crack faces in the j-direction by a point force pair in the $\pm i$-directions acting at location ρ, ϕ on the crack faces. The mode one tensile case crack face Green's function D_{yy} was derived by Gao and Rice (1987) to a closed expression (eqns (A-9) and (A-11) of their paper). We can also immediately verify that $D_{yr} = D_{y\theta} = 0$. The integrals involved in the shear mode crack face Green's functions seem formidable to integrate analytically.

Equation (6), or eqn (8) combined with eqn (9) gives the formulae for the relative crack surface displacement for a perfectly circular crack when the loading system $p_i(r,\theta)$ $(i = r, y, \theta)$ and the crack face weight functions are known.

Now let us consider a slightly non-circular crack the tip of which is located at $r = a(\theta)$, where the function $a(\theta)$ differs modestly from a constant. For the purpose of retaining the correct asymptotic behavior near the crack front at an angle θ, we shall take the reference crack front to be a circle of radius $a(\theta)$ so that we are able to let r approach simultaneously the reference front and the actual perturbed front along the ray at any particular chosen angle θ. As described by Rice (1985a) and Gao and Rice (1986, 1987), such a relocation of the reference crack front when θ changes is crucial for the calculation of the stress intensity factors along the perturbed crack front. In other words, the reference front has to be relocated as above to make eqn (1) valid even in the vicinity of the actual crack front. For conciseness in presentation of formulae, we let a stand for $a(\theta)$ from now on and note that $\delta a(\theta') = a(\theta') - a(\theta)$ at location θ' along the crack front in this arrangement. Therefore, eqn (4) becomes

$$\delta[\Delta u_j(r,\theta)] = 2\frac{(1-\nu^2)}{E}\int_0^{2\pi}\left(\sum_{\alpha=1}^{3} A_\alpha K_\alpha^0[\theta';a]k_{\alpha j}(\theta';r,\theta;a)\right)[a(\theta')-a]a\,\mathrm{d}\theta' \tag{10}$$

where $a = a(\theta)$ should be kept in mind. Equation (10) plus eqn (6) then gives the total relative displacement for a nearly circular crack

$$\begin{aligned}\Delta u_j(r,\theta) &= \Delta u_j^0[r,\theta;a] + \delta[\Delta u_j(r,\theta)]\\ &= 2\frac{(1-\nu^2)}{E}\int_0^{2\pi}\left\{\int_r^a \sum_{\alpha=1}^{3} A_\alpha K_\alpha^0[\theta';a']k_{\alpha j}(\theta';r,\theta;a')a'\,\mathrm{d}a'\right.\\ &\quad\left.+\left(\sum_{\alpha=1}^{3} A_\alpha K_\alpha^0[\theta';a]k_{\alpha j}(\theta';r,\theta;a)\right)[a(\theta')-a]\right\}a\,\mathrm{d}\theta'\\ &\simeq 2\frac{(1-\nu^2)}{E}\int_r^{a(\theta')}\int_0^{2\pi}\left(\sum_{\alpha=1}^{3} A_\alpha K_\alpha^0[\theta';a']k_{\alpha j}(\theta';r,\theta;a')\right)a'\,\mathrm{d}\theta'\,\mathrm{d}a'\end{aligned} \tag{11}$$

where the $\simeq$ sign means equal to first-order accuracy. Everything here is again exact to first-order accuracy in $a(\theta') - a$. Equation (11) plus eqn (7) then enables us to calculate the relative crack surface displacement for a nearly circular crack when the loading profile $p_i(\rho,\phi)$ and the actual shape function $a(\theta)$ are known.

CRACK FACE WEIGHT FUNCTIONS

From the above discussion, we know that knowledge of the crack face weight function $k_{\alpha j}$ for a perfectly circular reference crack is necessary for the calculation of the relative crack surface displacement for a nearly circular crack. Fortunately, those functions $k_{\alpha j}$ were derived by Bueckner (1987). We present them here in our coordinate system and notations

$$\begin{aligned}
k_{1y}(\theta';r,\theta;a) &= k = \frac{\sqrt{((a^2-r^2)/a\pi^3)}}{d^2}\\
k_{2r}(\theta';r,\theta;a) &= -\frac{k}{2-\nu}\left\{\nu\frac{d^2}{ar}+2\frac{d}{r}\cos\lambda+\frac{a}{r}[\nu-2-2\nu\cos 2\lambda]\right\}\\
k_{3r}(\theta';r,\theta;a) &= \frac{2k}{2-\nu}\left\{\frac{d}{r}\sin\lambda-\nu\frac{a}{r}\sin 2\lambda\right\}\\
k_{2\theta}(\theta';r,\theta;a) &= -\frac{2k}{2-\nu}\left\{(1-\nu)\frac{d}{r}\sin\lambda+\nu\frac{a}{r}\sin 2\lambda\right\}\\
k_{3\theta}(\theta';r,\theta;a) &= \frac{k}{2-\nu}\left\{\nu\frac{d^2}{ar}-2(1-\nu)\frac{d}{r}\cos\lambda+\frac{a}{r}[2-\nu-2\nu\cos 2\lambda]\right\}\\
k_{2y}(\theta';r,\theta;a) &= k_{3y} = k_{1\theta} = k_{1r} = 0
\end{aligned} \tag{12}$$

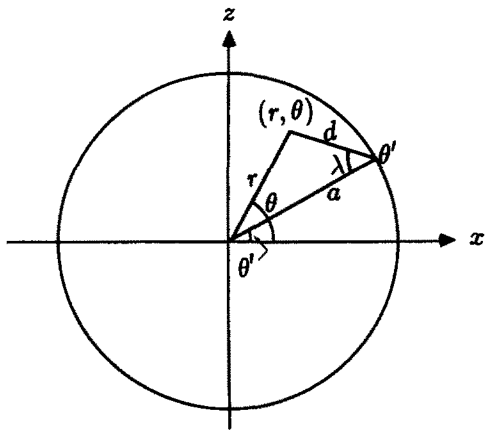

Fig. 1. A perfectly circular crack on $y = 0$ in an infinite elastic body; θ' denotes the location along the crack front.

where $d^2 = a^2+r^2-2ar\cos(\theta'-\theta)$ is the square of the distance between the point r, θ and the point a, θ' along the crack front, and λ is an angle shown in Fig. 1, which has the same sign as $\theta-\theta'$ with $|\theta-\theta'| < \pi$. We also have the following supplementary geometrical relations referring to Fig. 1

$$\begin{aligned} \cos\lambda &= [a-r\cos(\theta'-\theta)]/d; \quad \sin\lambda = -r\sin(\theta'-\theta)/d \\ \cos 2\lambda &= 1-2r^2\sin^2(\theta'-\theta)/d^2; \quad \sin 2\lambda = -2[a-r\cos(\theta'-\theta)]r\sin(\theta'-\theta)/d^2. \end{aligned} \tag{13}$$

Equations (12) and (13) provide a full definition of the crack face weight functions for an internal circular crack. These equations are mathematically well defined everywhere except at $r = 0$ where eqns (12) are not directly evaluable. Nevertheless one can show, by taking the limit $r \to 0$ in eqns (12) and (13), that

$$\begin{aligned} & k_{1y}(\theta';0,\theta;a) = k = 1/(a\pi)^{3/2}, && k_{2r}(\theta';0,\theta;a) = \frac{2k(1+\nu)}{2-\nu}\cos(\theta'-\theta) \\ & k_{3r}(\theta';0,\theta;a) = -\frac{2k(1-2\nu)}{2-\nu}\sin(\theta'-\theta), && k_{2\theta}(\theta';0,\theta;a) = \frac{2k(1+\nu)}{2-\nu}\sin(\theta'-\theta) \\ & k_{3\theta}(\theta';0,\theta;a) = \frac{2k(1-2\nu)}{2-\nu}\cos(\theta'-\theta), && k_{2y}(\theta';0,\theta;a) = k_{3y} = k_{1\theta} = k_{1r} = 0. \end{aligned} \tag{14}$$

Kassir and Sih (1975) presented solutions for intensity factors due to a point force pair in the radial direction, i.e. k_{2r} and k_{3r}. Their solutions are not in agreement with the corresponding solutions in eqns (12). Therefore, it is necessary to check the validity of eqns (12) and (13). One may observe that in the limit $a \to \infty$, the circular crack becomes a half-plane crack. Therefore, eqns (12), in the same limit, should approach the corresponding solutions for crack face weight functions for a half-plane crack. Assuming in that limit the crack front lies along the z-axis and $x < 0$ denotes the crack face, the polar coordinates r, θ in the crack plane are replaced by Cartesian coordinates x, z in the following manner:

$$r-a \to x; \quad a\theta \to -z. \tag{15}$$

Using the following asymptotic relations when $a \to \infty$

$$
\begin{aligned}
& a-r=-x; \quad d \sin \lambda = z'-z; \quad d \cos \lambda = -x \\
& d^2 \sin 2\lambda = -2x(z'-z); \quad d^2 \cos 2\lambda = x^2-(z'-z)^2
\end{aligned} \tag{16}
$$

where now $d^2 = x^2+(z'-z)^2$ is the square of the distance between a point x, z on the crack face and a point $0, z'$ along the crack front. It can be shown that eqns (12) are reduced to

$$
\begin{aligned}
k_{1y} &= k = \frac{(-2x/\pi^3)^{1/2}}{[x^2+(z'-z)^2]}, \quad k_{1x} = k_{1z} = k_{2y} = k_{3y} = 0 \\
k_{2r} &= k_{2x} = \left[1+\frac{2\nu}{2-\nu}\frac{x^2-(z'-z)^2}{x^2+(z'-z)^2}\right]k \\
k_{3\theta} &= k_{3z} = \left[1-\frac{2\nu}{2-\nu}\frac{x^2-(z'-z)^2}{x^2+(z'-z)^2}\right]k \\
-k_{2\theta} &= k_{2z} = -\frac{4\nu}{2-\nu}\frac{x(z'-z)}{x^2+(z'-z)^2}k \\
-k_{3r} &= k_{3x} = -\frac{4\nu}{2-\nu}\frac{x(z'-z)}{x^2+(z'-z)^2}k.
\end{aligned} \tag{17}
$$

Equations (17) match the correct point force intensity factor formulae for a half-plane crack (see, e.g. Tada *et al.* (1973)). Although the solutions proposed by Kassir and Sih (1975) do give the correct solutions when $r = 0$, they failed to match the above correct formulae for a half-plane crack in the limit that the radius of the crack a approaches to infinity. In fact, their solutions suggest that $k_{3r} \to 0$ and $k_{2r} \to \infty$ when $a \to \infty$.

SHEAR STRESS INTENSITY FACTORS

Let us consider a slightly non-circular crack the shape of which is described by the function $a(\theta)$ which differs modestly from a constant. As indicated by formulae (2), very near the crack front the stress intensity factors are asymptotically proportional to the crack face relative displacement. Under the present coordinate system that relation becomes

$$
\begin{aligned}
\Delta u_n(r,\theta) &= \frac{8(1-\nu^2)}{E}\sqrt{\left(\frac{a-r}{2\pi}\right)}K_2(\theta)+O[(a-r)^{3/2}] \\
\Delta u_t(r,\theta) &= \frac{8(1+\nu)}{E}\sqrt{\left(\frac{a-r}{2\pi}\right)}K_3(\theta)+O[(a-r)^{3/2}]
\end{aligned} \tag{18}
$$

where n and t now are the normal and tangential directions along the slightly non-circular crack front with n lying in the r, θ plane (Fig. 2). The same relation would also hold between the variation of these quantities from the reference state in which the crack front is perfectly circular. Note that quantities $\Delta u_n(r,\theta)$ and $\Delta u_t(r,\theta)$ should be associated with the current directions n and t during the perturbation of the crack front.

Let us introduce the following asymptotic near tip expansion for the relative displacement variation for conciseness

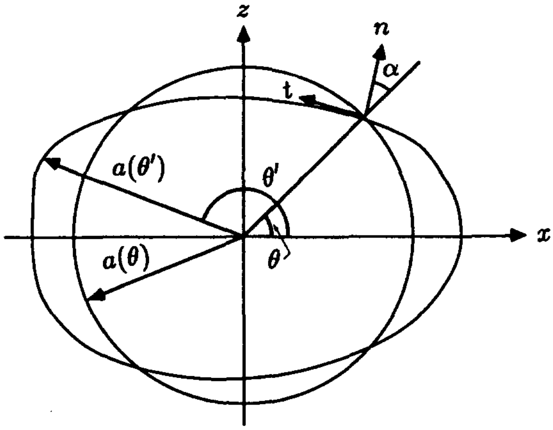

Fig. 2. Near circular crack in an infinite elastic body. Reference circular crack front $r = a(\theta)$ centered on $(0, 0)$; θ' denotes the location along the actual front and $\delta a(\theta') = a(\theta') - a(\theta)$ denotes the advance of the crack front location in the plane $y = 0$; normal (n) and tangential (t) directions at location θ along the actual crack front.

$$\begin{aligned}\delta[\Delta u_r(r,\theta)] &= \frac{8(1-\nu^2)}{E}\sqrt{\left(\frac{a-r}{2\pi}\right)}\delta P(\theta) + O[(a-r)^{3/2}] \\ \delta[\Delta u_\theta(r,\theta)] &= \frac{8(1+\nu)}{E}\sqrt{\left(\frac{a-r}{2\pi}\right)}\delta Q(\theta) + O[(a-r)^{3/2}]\end{aligned} \tag{19}$$

where

$$\begin{aligned}\delta P(\theta) &= \lim_{r\to a}\int_0^{2\pi}\sqrt{\left(\frac{\pi}{8(a-r)}\right)}\left\{\sum_{\alpha=1}^{3} A_\alpha K_\alpha^0[\theta';a]k_{\alpha r}(\theta';r,\theta;a)\right\}[a(\theta')-a]a\,\mathrm{d}\theta' \\ \delta Q(\theta) &= \lim_{r\to a}\int_0^{2\pi}(1-\nu)\sqrt{\left(\frac{\pi}{8(a-r)}\right)}\left\{\sum_{\alpha=1}^{3} A_\alpha K_\alpha^0[\theta';a]k_{\alpha\theta}(\theta';r,\theta;a)\right\}[a(\theta')-a]a\,\mathrm{d}\theta'.\end{aligned} \tag{20}$$

Since $k_{1r} = k_{1\theta} = 0$ the summation over α in the above equations is actually from 2 to 3. It will be shown that eqns (20), together with eqns (12), can be finally reduced to

$$\begin{aligned}\delta P(\theta) = \frac{1}{2\pi a(2-\nu)}PV\int_0^{2\pi}&\left\{\frac{2\cos(\theta'-\theta)-3\nu}{4\sin^2[(\theta'-\theta)/2]}K_2^0[\theta';a]\right. \\ &\left.-\cot\left(\frac{\theta'-\theta}{2}\right)K_3^0[\theta';a]\right\}[a(\theta')-a]\,\mathrm{d}\theta' + \frac{\nu}{(2-\nu)(1-\nu)}K_3^0[\theta';a]\frac{\mathrm{d}a(\theta)}{a\,\mathrm{d}\theta} \\ \delta Q(\theta) = \frac{1}{2\pi a(2-\nu)}PV\int_0^{2\pi}&\left\{\frac{2(1-\nu)\cos(\theta'-\theta)+3\nu}{4\sin^2[(\theta'-\theta)/2]}K_3^0[\theta';a]\right. \\ &\left.+(1-\nu)\cot\left(\frac{\theta'-\theta}{2}\right)K_2^0[\theta';a]\right\}[a(\theta')-a]\,\mathrm{d}\theta' + \frac{\nu(1-\nu)}{(2-\nu)}K_2^0[\theta';a]\frac{\mathrm{d}a(\theta)}{a\,\mathrm{d}\theta}\end{aligned} \tag{21}$$

to first order in $a(\theta') - a$ and in $\mathrm{d}a(\theta)/\mathrm{d}\theta$, where PV in eqns (21) denotes the principal value in the Cauchy sense. When substituting these equations into eqns (19) we get the changes in relative surface displacement at a point r, θ very near the crack front in going from the hypothetical reference state, in which the crack front is perfectly circular and of radius $a = a(\theta)$, to the actual state in which the crack front differs modestly from a circle.

Equations (21) can be proved as follows. Breaking the integrals $\int_0^{2\pi}$ in eqns (20) into $\int_0^{\theta-\eta}+\int_{\theta-\eta}^{\theta+\eta}+\int_{\theta+\eta}^{2\pi}$, the $\int_0^{\theta-\eta}+\int_{\theta+\eta}^{2\pi}$ parts, when letting $r\to a$ (remember $a=a(\theta)$) and then $\eta\to 0^+$, gives the PV term in eqns (21) above, whereas the remaining $\int_{\theta-\eta}^{\theta+\eta}$ part of $\delta P(\theta)$ becomes after taking the variable transformation $\theta'-\theta=t$

$$-\frac{1}{2\pi(2-\nu)}\lim_{r\to a^-}\int_{-\eta}^{\eta}\frac{K_2^0[t+\theta;a]}{d^2}\left\{\nu\frac{d^2}{ar}+2\frac{d}{r}\cos\lambda+\frac{a}{r}[\nu-2-2\nu\cos 2\lambda]\right\}[a(\theta')-a]\,\mathrm{d}\theta'$$

$$+\frac{1}{\pi(2-\nu)(1-\nu)}\lim_{r\to a^-}\int_{-\eta}^{\eta}\frac{K_3^0[t+\theta;a]}{d^2}\left\{\frac{d}{r}\sin\lambda-\nu\frac{a}{r}\sin 2\lambda\right\}[a(\theta')-a]\,\mathrm{d}\theta' \quad (22)$$

where $d^2=a^2+r^2-2ar\cos t$ and from eqns (13)

$$\begin{aligned}\cos\lambda&=(a-r\cos t)/d;\quad \sin\lambda=-r\sin t/d\\ \cos 2\lambda&=1-2r^2\sin^2 t/d^2;\quad \sin 2\lambda=-2(a-r\cos t)r\sin t/d^2.\end{aligned} \quad (23)$$

Now let us observe that

$$K_\alpha^0[t+\theta;a][a(t+\theta)-a]=K_\alpha^0[\theta;a]\frac{\mathrm{d}a(\theta)}{\mathrm{d}\theta}t+O[t^2];\quad (a=a(\theta)) \quad (24)$$

and that the error term $O[t^2]$ will have a bound of form $|O[t^2]|\leqslant Bt^2$ on $-\eta\leqslant t\leqslant\eta$ for some finite $B>0$. The term linear in t gives a zero contribution to the first integral in eqn (22), i.e. to the integral involving K_2^0 because the rest of the integrand forms an even function of t by eqns (23). It can also be justified that when $r\to a^-$

$$|\nu d^2/(ar)+2(d/r)\cos\lambda+(a/r)[\nu-2-2\nu\cos 2\lambda]|\leqslant C$$

for some finite $C\geqslant 0$ and that $d^2=(a-r)^2+4ar\sin^2[t/2]\geqslant 4ar\sin^2[t/2]\sim art^2$ when $-\eta\leqslant t\leqslant\eta$ for small η. We thus have

$$\left|\frac{1}{2\pi(2-\nu)}\lim_{r\to a^-}\int_{-\eta}^{\eta}\frac{K_2^0[t+\theta;a]}{d^2}\left\{\nu\frac{d^2}{ar}+2\frac{d}{r}\cos\lambda+\frac{a}{r}[\nu-2-2\nu\cos 2\lambda]\right\}[a(\theta')-a]\,\mathrm{d}\theta'\right|$$

$$\leqslant\frac{BC}{2a\pi(2-\nu)}\int_{-\eta}^{\eta}\frac{t^2}{4\sin^2[t/2]}\,\mathrm{d}t\sim\frac{BC\eta}{a\pi(2-\nu)}. \quad (25)$$

Hence the upper bound on the first integral, and therefore the integral itself vanishes when letting $\eta\to 0$. It could be further shown by using eqns (23) that in the same limits, the second integral in eqn (22), involving K_3^0, becomes with the substitution $at=(a-r)\rho$

$$\frac{2\nu}{\pi(2-\nu)(1-\nu)}\int_{-\infty}^{\infty}\frac{\rho^2}{(1+\rho^2)^2}\,\mathrm{d}\rho\;K_3^0[\theta;a]\frac{\mathrm{d}a(\theta)}{a\,\mathrm{d}\theta}=\frac{\nu}{(2-\nu)(1-\nu)}K_3^0[\theta;a]\frac{\mathrm{d}a(\theta)}{a\,\mathrm{d}\theta}. \quad (26)$$

Equation $(21)_2$ can be similarly derived following the same steps.

In the above argument we have implicitly assumed, in writing the error terms as $O[t^2]$, that $K_\alpha^0[\theta';a][a(\theta')-a]$ has a good second derivative at θ. However, the steps leading to eqns (21) above may still be justified under the weaker assumption that the first derivative of $K_\alpha^0[\theta';a][a(\theta')-a]$ exists and is merely Holder continuous at θ, such that the bounded term above may be written as $B|t|^{1+\varepsilon}$ where $0<\varepsilon\leqslant 1$.

The variation of surface displacements contains two contributions. The PV term from $\int_0^{\theta-\eta}+\int_{\theta+\eta}^{2\pi}$ $(\eta\to 0^+)$, which represents the influence of the rest of the non-circular crack front on the relative displacement near the special point a,θ along the actual crack front, conveniently named as the "global effect", and the term from $\int_{\theta-\eta}^{\theta+\eta}$ which represents the contributions due to local slope change in the perturbation process, or the "local effect" for shortness. Similar comments were made in an earlier work (Gao and Rice, 1986) on the shear stress intensity factor for a slightly non-straight half-plane crack. Comparing with the solutions for a half-plane crack we observe that the coupling terms between mode 2 and mode 3 fracturing are now found in both the global effect and the local effect while in the case of a half-plane crack they only exist in the local effect. In fact it can be seen that when $a \to \infty$ the coupling term in PV integrals vanishes, but the coupling terms in the local effect remain. In fact, if we rewrite $a\theta$ as arc length position s along the reference front, the local effects of $\delta P(\theta)$ and $\delta Q(\theta)$ are

$$\nu K_3^0[s\,;a(s)]/[(2-\nu)\,(1-\nu)]\,[\mathrm{d}a(s)/\mathrm{d}s] \quad \text{and} \quad \nu(1-\nu)K_2^0[s\,;a(s)]/(2-\nu)\,[\mathrm{d}a(s)/\mathrm{d}s] \tag{27}$$

respectively. They have exactly the same form as those for a half-plane crack (Gao and Rice, 1986). This is not unexpected because the limiting process in calculating the local effect is equivalent to stretching the small differential segment of the crack front containing point s infinitely to a half-plane crack. More intuitively it can be imagined that in the very near neighborhood of some special point s along the circular crack front, one would not be able to tell whether the whole crack front is a straight line or a circle of some finite radius. The above argument remains valid even for an arbitrary, smoothly curved crack. For this reason, expressions (27) will be generally valid for the local slope effect on the variation of relative crack surface displacements when any smoothly curved crack front gets perturbed with a slope change $\mathrm{d}a(s)/\mathrm{d}s$ at location s along the crack front.

Referring to Fig. 2, we can get the relative displacement components in the normal and tangential directions along the crack front in terms of $\Delta u_r(r,\theta)$ and $\Delta u_\theta(r,\theta)$

$$\begin{aligned}\Delta u_n(r,\theta) &= \Delta u_r(r,\theta)\cos\alpha - \Delta u_\theta(r,\theta)\sin\alpha \\ \Delta u_t(r,\theta) &= \Delta u_r(r,\theta)\sin\alpha + \Delta u_\theta(r,\theta)\cos\alpha\end{aligned} \tag{28}$$

where α is the angle between the normal n- and r-axis. In fact $\tan\alpha = \mathrm{d}a(\theta)/(a\,\mathrm{d}\theta)$. Therefore, as α is small for a small perturbation, we have to first order

$$\begin{aligned}\Delta u_n(r,\theta) &= \Delta u_r - \Delta u_\theta \frac{\mathrm{d}a(\theta)}{a\,\mathrm{d}\theta} \\ \Delta u_t(r,\theta) &= \Delta u_\theta + \Delta u_r \frac{\mathrm{d}a(\theta)}{\mathrm{d}\theta}.\end{aligned} \tag{29}$$

We now write $\Delta u_r(\theta)$ as $\Delta u_r^0[\theta\,;a]+\delta[\Delta u_r(\theta)]$ and $\Delta u_\theta(\theta)$ as $\Delta u_\theta^0[\theta\,;a]+\delta[\Delta u_\theta(\theta)]$, i.e. the sum of near-tip relative crack surface displacements in the reference circular crack front configuration and the variations of relative crack surface displacement due to the crack front being perturbed from a circle, i.e. due to the crack front advancing by $\delta a(\theta') = a(\theta')-a$. We then have

$$\begin{aligned}\Delta u_n(r,\theta) &= \Delta u_r^0[\theta\,;a]+\delta[\Delta u_r(\theta)] - \Delta u_\theta^0[\theta\,;a]\frac{\mathrm{d}a(\theta)}{a\,\mathrm{d}\theta} \\ \Delta u_t(r,\theta) &= \Delta u_\theta^0[\theta\,;a]+\delta[\Delta u_\theta(\theta)] + \Delta u_r^0[\theta\,;a]\frac{\mathrm{d}a(\theta)}{a\,\mathrm{d}\theta}.\end{aligned} \tag{30}$$

Everything here is exact to first-order accuracy in the deviation of the actual crack front from a circle of radius $a = a(\theta)$. Comparing these expressions, as evaluated with the help of eqns (19) and (21), to eqns (18) we obtain the stress intensity factors K_2 and K_3 to first order when the crack front deviates from a reference circular front. The results, supplemented for completeness with the result for the mode 1 stress intensity factor derived by Gao and Rice (1987), are as follows:

$$K_1(\theta) = K_1^0[\theta;a(\theta)] + \frac{1}{8\pi} PV \int_0^{2\pi} \frac{K_1^0[\theta';a(\theta)]\,[a(\theta')/a(\theta)-1]}{\sin^2[(\theta'-\theta)/2]}\,d\theta'$$

$$\begin{aligned} K_2(\theta) = {} & K_2^0[\theta;a(\theta)] - \frac{2}{2-\nu} K_3^0[\theta;a(\theta)] \frac{1}{a(\theta)} \frac{da(\theta)}{d\theta} + \frac{1}{2\pi(2-\nu)} \\ & \times PV \int_0^{2\pi} \left\{ K_2^0[\theta';a(\theta)] \frac{2\cos(\theta'-\theta)-3\nu}{4\sin^2[(\theta'-\theta)/2]} - K_3^0[\theta';a(\theta)] \cot\left(\frac{\theta'-\theta}{2}\right) \right\} \\ & \times [a(\theta')/a(\theta)-1]\,d\theta' \end{aligned} \tag{31}$$

$$\begin{aligned} K_3(\theta) = {} & K_3^0[\theta;a(\theta)] + \frac{2(1-\nu)}{2-\nu} K_2^0[\theta;a(\theta)] \frac{1}{a(\theta)} \frac{da(\theta)}{d\theta} + \frac{1}{2\pi(2-\nu)} \\ & \times PV \int_0^{2\pi} \left\{ K_3^0[\theta';a(\theta)] \frac{2(1-\nu)\cos(\theta'-\theta)+3\nu}{4\sin^2[(\theta'-\theta)/2]} \right. \\ & \left. + (1-\nu)\cot\left(\frac{\theta'-\theta}{2}\right) K_2^0[\theta';a(\theta)] \right\} [a(\theta')/a(\theta)-1]\,d\theta'. \end{aligned}$$

It can be easily shown that eqns (31), in the limit when $a \to \infty$, reduce to the corresponding first-order shear stress intensity factor formulae for a half-plane crack derived by Gao and Rice (1986).

AXISYMMETRIC LOADS; HARMONIC WAVE FORM PERTURBATIONS

Now consider, for example, the case when the crack system is subjected to axisymmetric loading which consists of some distribution of radial and/or tangential forces so that the stress intensity factors thus induced by a given load system are independent of the location along the circular crack front, i.e. $K_\alpha^0[\theta;a] = K_\alpha^0[a]$ ($\alpha = 1,2,3$). Therefore, eqns (31) become

$$K_\alpha(\theta) = K_\alpha^0[a(\theta)] + \sum_{\beta=1}^{3} C_{\alpha\beta} K_\beta^0[a(\theta)] \tag{32}$$

for $\alpha, \beta = 1, 2, 3$, where coefficients $C_{\alpha\beta}$ have been introduced as

$$\begin{aligned} C_{11} &= \frac{1}{2\pi} PV \int_0^{2\pi} \frac{1}{4\sin^2[(\theta'-\theta)/2]} [a(\theta')/a(\theta)-1]\,d\theta' \\ C_{22} &= \frac{1}{2\pi(2-\nu)} PV \int_0^{2\pi} \frac{2\cos(\theta'-\theta)-3\nu}{4\sin^2[(\theta'-\theta)/2]} [a(\theta')/a(\theta)-1]\,d\theta' \\ C_{23} &= -\frac{1}{2-\nu} \left\{ \frac{2}{a(\theta)} \frac{da(\theta)}{d\theta} + \frac{1}{2\pi} PV \int_0^{2\pi} \cot[(\theta'-\theta)/2]\,[a(\theta')/a(\theta)-1]\,d\theta' \right\} \\ C_{33} &= \frac{1}{2\pi(2-\nu)} PV \int_0^{2\pi} \frac{2(1-\nu)\cos(\theta'-\theta)+3\nu}{4\sin^2[(\theta'-\theta)/2]} [a(\theta')/a(\theta)-1]\,d\theta' \\ C_{32} &= -(1-\nu)C_{23}; \quad C_{\alpha\beta} = 0, \quad \text{otherwise.} \end{aligned} \tag{33}$$

Note that $C_{\alpha\beta}$ ($\alpha, \beta = 1,2,3$) forms a 3×3 matrix, which when multiplied by the intensity factors for a reference circular crack under a given load system gives the variation of stress intensity factors due to the perturbation $a(\theta')-a(\theta)$. Let $C_{\alpha\beta}$ be called the "perturbation matrix", which has the same order of magnitude as $a(\theta')-a(\theta)$ and $da(\theta)/d\theta$. The results of eqn (32) for $K_\alpha(\theta)$ can be used to compute the energy release rate $G(\theta)$ along the slightly non-circular front by the relation

$$G = (1-\nu^2)(K_1^2+K_2^2)/E+(1+\nu)K_3^2/E \tag{34}$$

where E is Young's modulus and ν Poisson's ratio. When the calculation is done and we retain only those terms consistent with first-order accuracy in $\max_{0<\theta'<2\pi}|a(\theta')-a(\theta)|$ and $\mathrm{d}a(\theta)/\mathrm{d}\theta$, we find with the help of eqns (33) that the coupling terms involving $K_2^0K_3^0$ cancel one another so that

$$G(\theta) = G^0[a(\theta)]+2\frac{(1-\nu^2)}{E}\{C_{11}(K_1^0[a(\theta)])^2+C_{22}(K_2^0[a(\theta)])^2+C_{33}(K_3[a(\theta)])^2/(1-\nu)\}. \tag{35}$$

This phenomenon was also observed in an earlier paper (Gao and Rice, 1986).

Specifically, let us consider a crack with the harmonic wave form perturbation

$$a(\theta) = a_0+\mathrm{Re}\,[A\,\mathrm{e}^{in\theta}] \tag{36}$$

where a_0 is a real constant, n is an integer which represents the number of waves on the circumference of the circle $r = a_0$, A is a constant (possibly complex) and $|A|/a_0 \ll 1$. The notations Re $[c]$ and Im $[c]$ will be used to denote the real and imaginary parts of some arbitrary complex number c. We can also define the wavelength L of the perturbation profile so that $L = 2\pi a_0/n$. Substituting eqn (36) into eqns (33), and discarding terms of higher than first-order small terms, we have

$$C_{11} = -\frac{n}{2a_0}\mathrm{Re}\,[A\,\mathrm{e}^{in\theta}];\quad C_{23} = \frac{2n+1}{(2-\nu)a_0}\mathrm{Im}\,[A\,\mathrm{e}^{in\theta}];\quad C_{32} = -(1-\nu)C_{23};$$

$$C_{22} = -\frac{(2-3\nu)n-2}{2(2-\nu)a_0}\mathrm{Re}\,[A\,\mathrm{e}^{in\theta}];\quad C_{33} = -\frac{(2+\nu)n-2(1-\nu)}{2(2-\nu)a_0}\mathrm{Re}\,[A\,\mathrm{e}^{in\theta}].$$

Therefore, eqn (32) becomes

$$\begin{aligned}
K_1(\theta) &= K_1^0[a_0]+\left(\frac{\mathrm{d}K_1^0[a_0]}{\mathrm{d}a_0}-\frac{n}{2a_0}K_1^0[a_0]\right)\mathrm{Re}\,[A\,\mathrm{e}^{in\theta}];\\
K_2(\theta) &= K_2^0[a_0]+\left(\frac{\mathrm{d}K_2^0[a_0]}{\mathrm{d}a_0}-\frac{(2-3\nu)n-2}{(2-\nu)2a_0}K_2^0[a_0]\right)\mathrm{Re}\,[A\,\mathrm{e}^{in\theta}]\\
&\quad-\frac{2n+1}{(2-\nu)a_0}K_3^0[a_0]\,\mathrm{Im}\,[A\,\mathrm{e}^{in\theta}];\\
K_3(\theta) &= K_3^0[a_0]+\left(\frac{\mathrm{d}K_3^0[a_0]}{\mathrm{d}a_0}-\frac{(2+\nu)n-2(1-\nu)}{(2-\nu)2a_0}K_3^0[a_0]\right)\mathrm{Re}\,[A\,\mathrm{e}^{in\theta}]\\
&\quad+\frac{(2n+1)(1-\nu)}{(2-\nu)a_0}K_2^0[a_0]\,\mathrm{Im}\,[A\,\mathrm{e}^{in\theta}];
\end{aligned} \tag{37}$$

and eqn (35) becomes

$$G(\theta) = G^0[a_0]+\left(\frac{\mathrm{d}G^0[a_0]}{\mathrm{d}a_0}-\frac{n}{a_0}F[a_0]\right)\mathrm{Re}\,[A\,\mathrm{e}^{in\theta}]. \tag{38}$$

Here

$$G^0[a_0] = \frac{1-\nu^2}{E}\left\{(K_1^0[a_0])^2+(K_2^0[a_0])^2+\frac{1}{1-\nu}(K_3^0[a_0])^2\right\} \tag{39}$$

and

$$F[a_0] = \frac{1-\nu^2}{E}\left\{(K_1^0[a_0])^2 + \left[\frac{2-3\nu}{2-\nu} - \frac{2}{2-\nu}\frac{L}{2\pi a_0}\right](K_2^0[a_0])^2 + \left[\frac{2+\nu}{(2-\nu)} - \frac{2(1-\nu)}{(2-\nu)}\frac{L}{2\pi a_0}\right](K_3^0[a_0])^2/(1-\nu)\right\}. \quad (40)$$

Note that the extrema of $G(\theta)$ do coincide with the extrema of $a(\theta)$, i.e. where $A\,e^{in\theta} = |A|$ since the coupling terms which contain terms of Im $[A\,e^{in\theta}]$ in eqn (40) have canceled one another.

We have not been able to find an energetic interpretation for F in eqn (38). The energy release rate G is sometimes called the "crack extension force" since it is the generalized force conjugate to crack growth. We assume that the quasi-static growth rate of the crack increases with this G at the same location along the front. Then a small harmonic perturbation of wave number n can be said to be configurationally stable during subcritical crack growth if the energetical force $G(\theta)$ is decreased from $G^0[\theta; a_0]$ when $a(\theta)$ exceeds a_0 and increased when $a(\theta)$ is less than a_0 and configurationally unstable if the opposite is true. Although the stability issue is less readily addressed under general mixed mode loadings as we have analyzed them here since a mixed mode crack will seldom grow along a plane, one case which may meet that condition of planarity involves the tectonic shear crack the slip surface of which is channeled by a pre-existing fault plane. In that case or other appropriate cases, we have the following stability condition:

$$dG^0[a_0]/da_0 < nF[a_0]/a_0. \quad (41)$$

This reduces to the formulae given by Gao and Rice (1987) for pure mode 1 conditions. The crack growth is then likely to amplify the forms of those unstable wave configurations, i.e. those which do not satisfy inequality (41). Of course, the growth or decay of the harmonic perturbations is understood to be superposed on the overall increment of a_0 in describing the total crack growth. Now consider some axisymmetrically distributed loading $p_j(r)$ $(j = x, y, z)$ on the crack faces. The following preliminary relations may be shown:

$$\begin{aligned} \int_0^{2\pi} k_{3r}(\theta'; \rho, \phi; a)\,d\phi &= \int_0^{2\pi} k_{2\theta}(\theta'; \rho, \phi; a)\,d\phi = \frac{2}{\sqrt{(\pi a^3)}}\frac{\rho}{\sqrt{(a^2-\rho^2)}} \\ \int_0^{2\pi} k_{3\theta}(\theta'; \rho, \phi; a)\,d\phi &= \int_0^{2\pi} k_{2r}(\theta'; \rho, \phi; a)\,d\phi = 0. \end{aligned} \quad (42)$$

Therefore, carrying out the integration on ϕ in eqn (6) results in

$$\begin{aligned} K_2^0[a] &= \frac{2}{\sqrt{(\pi a^3)}}\int_0^a \frac{\rho^2 p_r(\rho)}{\sqrt{(a^2-\rho^2)}}\,d\rho \\ K_3^0[a] &= \frac{2}{\sqrt{(\pi a^3)}}\int_0^a \frac{\rho^2 p_\theta(\rho)}{\sqrt{(a^2-\rho^2)}}\,d\rho. \end{aligned} \quad (43)$$

The tangential loading p_θ is more interesting as one would meet such loading in a simple torsion test. As an illustration, let us assume that the only non-zero stress components induced on a plane $y = $ const by the external loading is of form $\tau_{y\theta} = -C\rho^m$ for some constant C and m in the absence of the crack (or far away from the crack). Therefore, by the well-known superposition argument we can equivalently consider that the crack system is subjected to some tangential loading $p_\theta(\rho) = C\rho^m$ on the crack faces. By eqns (43), we have

$$K_2^0[a] = 0$$
$$K_3^0[a] = \frac{2C}{\sqrt{(\pi a^3)}} \int_0^a \frac{\rho^{m+2}}{\sqrt{(a^2-\rho^2)}}\,\mathrm{d}\rho = \frac{2Ca^m\sqrt{a}}{\sqrt{\pi}} \int_0^{\pi/2} \sin^{m+2} t\,\mathrm{d}t. \tag{44}$$

Equations (44) can then be written as

$$K_2^0[a] = 0\,; \quad K_3^0[a] = Ba^{m+1/2} \tag{45}$$

for some constant B. Note that when $m = 1$ eqns (45) are compatible with the corresponding result in Tada *et al.* (1973). Therefore

$$G^0[a_0] = \frac{1+\nu}{E}(K_3^0[a_0])^2; \quad \mathrm{d}G^0/\mathrm{d}a_0 = (2m+1)G^0/a_0;$$
$$F[a_0] = \frac{2+\nu}{2-\nu}\left[1 - \frac{2(1-\nu)}{(2+\nu)n}\right]G^0[a_0]. \tag{46}$$

Substituting eqns (46) into inequality (41) we have the stability condition

$$n > \frac{2-\nu}{2+\nu}(2m+1) + \frac{2(1-\nu)}{2+\nu} \tag{47}$$

when $\nu = 0.25$, inequality (47) reduces to

$$n > (14m+13)/9. \tag{48}$$

In a simple torsion test where a constant torque is applied to a linear elastic specimen, the shear stress in the absence of a crack is linearly increasing with r, therefore corresponds to the $m = 1$ case discussed above. Hence by inequality (48) the stability condition is $n > 3$ so that modes $n = 1, 2$ are unstable and mode $n = 3$ is neutrally stable. This suggests a strong irregularity in the growth of a circular crack under torsion test.

EQUILIBRIUM CRACK SHAPE UNDER UNIFORM FAR FIELD SHEAR LOADING

In the last section we studied how the stress intensity factors change when the crack front is perturbed from the reference circle for a crack system under axisymmetrical loading. It would be equally interesting to consider such a crack subjected to some complicated loading system that induces some varying K_2^0 and K_3^0 along the reference crack front. In that case it is hard to address the configurational stability problem of the planar crack since the intensity factors are not even uniform along the reference crack front. However, we could address the following: since in perturbing the crack front from a reference circle to another shape the distribution of stress intensity factors varies correspondingly, is it possible by slightly perturbing the crack front from its reference state to an "equilibrium" shape such that the energy release rate G, or some other characteristic quantities, distributes uniformly along the perturbed crack front? Because of its conjugation with the crack growth δa, the energy release rate, sometimes called the crack extension force, is reasonably assumed to be a critical quantity which controls the crack growth. Therefore, we call some perturbed shape corresponding to G distributed uniformly along the crack front an "equilibrium" shape. Of course some other critical quantities can also be used here if the growth of a crack is associated with that quantity. For example the maximum shear stress intensity factor $S = \sqrt{(K_2^2 + K_3^2)}$ can be considered in a similar way and it will lead to a slightly different equilibrium shape. In the following we will find a way to calculate the proper wave form perturbations of a circular crack that lead to the equilibrium shape.

The equilibrium shape of a planar crack under some mixed mode loading system is assumed here to be close to a circle in consistency with the first-order accurate perturbation analysis. This is not generally true, but we only consider the relevant cases. Such an equilibrium shape can be studied in the following manner: assume some function $a(\theta)$, which is dependent on the polar angle θ and is close to a constant, can be used to describe the equilibrium shape of the crack. We expand that shape function into a Fourier series and consider all terms other than the zeroth-order constant term as harmonic wave form perturbations. Therefore

$$a(\theta) = \sum_{k=-n}^{n} C_k \, e^{ik\theta} \tag{49}$$

where C_k $(k \neq 0)$ are some unknown complex constants to be determined and the series has been truncated to a finite series for approximation. The contributions to the energy release rate due to these perturbation terms can then be calculated from eqns (31) and then eqn (34) to first order in C_k $(k \neq 0)$ so that

$$G(\theta) = \sum_{k=-n}^{n} g_k \, e^{ik\theta} \tag{50}$$

where g_k is some complex constant coefficient that depends on C_i, $i = -n, \ldots, n$. Letting $g_k = 0$ for $k = -n, \ldots, n$ and $k \neq 0$, we then get $2n$ equations to solve for $2n$ unknown constants C_k in terms of C_0. Of course we can let $C_0 = 1$ at the beginning of the calculation in calculating the equilibrium shape. The actual magnitude of C_0 will be determined by the given load system.

As an example let us now consider the case when the crack system is subjected to a far field uniformly distributed shear loading τ in the $\theta = 0°$ direction. The stress intensity factors for a circular crack under such shear loads was given by Tada *et al.* (1973) as

$$K_2^0 = \alpha\sqrt{(\pi a)} \cos\theta; \quad K_3^0 = -(1-\nu)\alpha\sqrt{(\pi a)} \sin\theta \tag{51}$$

where $\alpha = 4\tau/[\pi(2-\nu)]$. As a first step let us consider an arbitrary harmonic wave form perturbation described by eqn (36) and substitute eqns (51) and (36) into eqns (31). Carrying out the principal value integrations we finally find that

$$\begin{aligned} K_2(\theta) &= K_2^0[\theta; a_0] + \frac{K_3^0[\theta; a_0]}{(1-\nu)a_0} \operatorname{Im}[A \, e^{i\theta}] \\ K_3(\theta) &= K_3^0[\theta; a_0] - \frac{K_2^0[\theta; a_0](1-\nu)}{a_0} \operatorname{Im}[A \, e^{i\theta}] \end{aligned} \tag{52}$$

for $n = 1$, and

$$\begin{aligned} K_2(\theta) &= K_2^0[\theta; a_0] - \left[\frac{2-3\nu}{2-\nu} n - \frac{2(1-\nu)}{2-\nu}\right] \frac{K_2^0[\theta; a_0]}{2a_0} \operatorname{Re}[A \, e^{in\theta}] \\ &\quad + \left[\frac{4}{2-\nu} n + \frac{\nu}{(1-\nu)(2-\nu)}\right] \frac{K_3^0[\theta; a_0]}{2a_0} \operatorname{Im}[A \, e^{in\theta}] \\ K_3(\theta) &= K_3^0[\theta; a_0] - \left[\frac{2+\nu}{2-\nu} n - \frac{2}{2-\nu}\right] \frac{K_3^0[\theta; a_0]}{2a_0} \operatorname{Re}[A \, e^{in\theta}] \\ &\quad - \left[\frac{4(1-\nu)}{2-\nu} n - \frac{\nu(1-\nu)}{2-\nu}\right] \frac{K_2^0[\theta; a_0]}{2a_0} \operatorname{Im}[A \, e^{in\theta}] \end{aligned} \tag{53}$$

for $n > 1$. Using the existing analytical solutions, eqns (52) and (53) are verified for the cases of $n = 1, 2$ in the Appendix. Using eqns (34), (52), and (53) to calculate the energy release rate G, we have

$$\frac{E}{1-\nu^2} G(\theta) = \frac{E}{1-\nu^2} G^0[\theta; a_0] + \frac{2\nu}{1-\nu} \frac{K_2^0[\theta; a_0] K_3^0[\theta; a_0]}{a_0} \text{Im} [A\, e^{i\theta}], \quad n = 1$$

$$\frac{E}{1-\nu^2} G(\theta) = \frac{E}{1-\nu^2} G^0[\theta; a_0] - \left[\frac{2-3\nu}{2-\nu} n - \frac{2(1-\nu)}{2-\nu}\right] \frac{(K_2^0[\theta; a_0])^2}{a_0} \text{Re} [A\, e^{in\theta}] \tag{54}$$

$$- \left[\frac{2+\nu}{2-\nu} n - \frac{2}{2-\nu}\right] \frac{(K_3^0[\theta; a_0])^2}{(1-\nu) a_0} \text{Re} [A\, e^{in\theta}] + \frac{\nu K_2^0[\theta; a_0] K_3^0[\theta; a_0]}{(1-\nu) a_0} \text{Im} [A\, e^{in\theta}], \quad n > 1$$

where now

$$\frac{E}{1-\nu^2} G^0[\theta; a_0] = \alpha^2 \pi a_0 [1 - \nu/2 + (\nu/2) \cos 2\theta]. \tag{55}$$

For most material $\nu = 0.25 \sim 0.30$. Therefore, $G^0[\theta; a_0]$ differs from a constant term only by a modest $\cos 2\theta$ dependent term. Therefore, one would conveniently expand the shape function into a cosine Fourier series as

$$a(\theta) = a_0 \left[1 + \sum_{n=2,4,\ldots} A_n \cos n\theta\right]. \tag{56}$$

Using eqns (54) and (56), we find that in this case

$$\frac{E}{1-\nu^2} \frac{G(\theta)}{\alpha^2 \pi a_0} = g_0 + g_2 \cos 2\theta + g_4 \cos 4\theta + \cdots \tag{57}$$

where

$$g_0 = 1 - \nu/2 + \frac{\nu}{4} A_2; \quad g_2 = \nu/2 - \frac{2 - 2\nu - \nu^2}{2-\nu} A_2 + \frac{3\nu}{4} A_4;$$
$$g_4 = \frac{3\nu}{4} A_2 - \frac{2(3 - 3\nu + \nu^2)}{2-\nu} A_4. \tag{58}$$

Letting the coefficient of $\cos n\theta$ vanish for $n = 2, 4$ after truncation of the above series, we see that A_2 is of order ν while A_4 is of order ν^2. Hence we drop the A_4 term to be consistent with the first-order analysis. Therefore, $A_2 = \nu(2-\nu)/[2(2-2\nu-\nu^2)] = 0.1522$ for $\nu = 0.25$. Hence the equilibrium shape of the crack with a uniform distribution of the crack extension force G is approximately

$$a(\theta) = a_0(1 + 0.1522 \cos 2\theta + \cdots). \tag{59}$$

This is very close to an ellipse. The above $a(\theta)$ is plotted in Fig. 3.

Consider now that the crack growth criteria are such that the crack grows only when the maximum shear stress intensity defined by $S(\theta) = [K_2(\theta)]^2 + [K_3(\theta)]^2$ attains a critical value. Again, it is analogous to see that

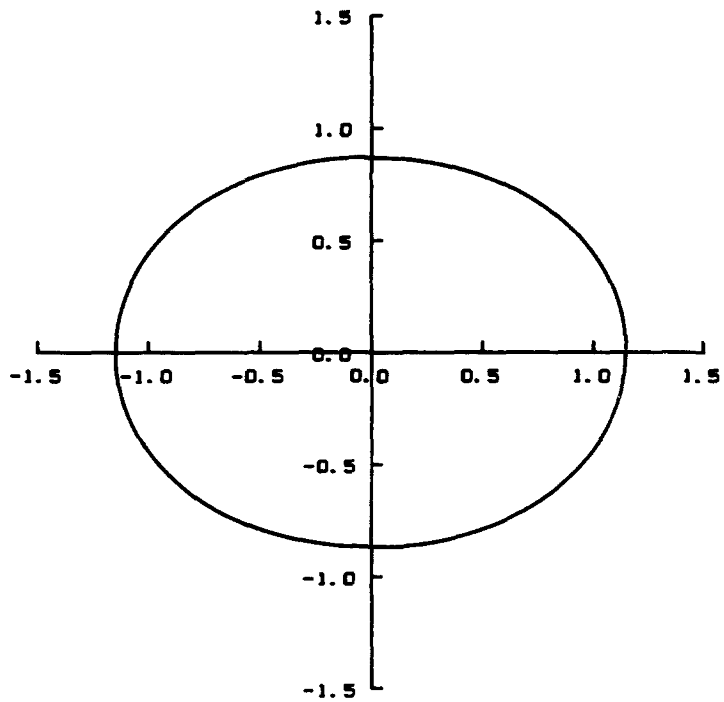

Fig. 3. The "equilibrium shape" of a planar crack under remotely uniform shear loading; corresponds to the uniform distribution of the crack extension force.

$$\begin{aligned} S^0[\theta;a_0] &= (K_2^0[\theta;a_0])^2+(K_3^0[\theta;a_0])^2 \\ &= \alpha^2\pi a_0[1-\nu(2-\nu)/2+(\nu/2)(2-\nu)\cos 2\theta]. \end{aligned} \tag{60}$$

Following the same steps leading to eqn (59), we have the equilibrium shape function, again for $\nu = 0.25$ as

$$a(\theta) = a_0(1+0.2913 \cos 2\theta+\cdots) \tag{61}$$

which is about twice as big as the amplitude of the previous perturbation.

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APPENDIX: PERTURBATION FORMULAE VERSUS ANALYTICAL SOLUTIONS

In the text we derived perturbation formulae of eqns (52) and (53) for the harmonic wave form perturbation $a(\theta) = a_0 + \mathrm{Re}\,[A\,e^{in\theta}]$. One may note that the above perturbation corresponds to a small translational shift of the circular crack for $n = 1$ and corresponds to slightly squeezing the circular shape of the unperturbed crack into an ellipse for $n = 2$. For both cases we know the exact solutions for the perturbed crack. Therefore, for the sake of confidence in eqns (52) and (53) we check them against the correct first-order behavior of analytical solutions in the two special cases, i.e. $n = 1, 2$.

We consider perturbations in the form of $a(\theta) = a_0 + A\cos n\theta$ for $A > 0$ without loss of generality. In the first case, $n = 1$, eqns (51) still apply to the perturbed crack if replacing θ by a new polar angle β associated with the perturbed center. We also have the geometrical relations

$$A + a_0\cos\beta = a(\theta)\cos\theta;\quad a_0\sin\beta = a(\theta)\sin\theta. \tag{A1}$$

It is then ready to show that

$$\begin{aligned} K_2 &= \alpha\sqrt{(\pi a_0)}\cos\beta = K_2^0[\theta;a_0] + \frac{K_3^0[\theta;a_0]}{(1-\nu)a_0}A\sin\theta \\ K_3 &= -(1-\nu)\alpha\sqrt{(\pi a_0)}\sin\beta = K_3^0[\theta;a_0] - \frac{K_2^0[\theta;a_0](1-\nu)}{a_0}A\sin\theta. \end{aligned} \tag{A2}$$

Hence eqns (52) match the correct analytical first-order formulae in eqns (A2).

For the $n = 2$ case, let us write the corresponding analytical solutions for an elliptical shear crack by Kassir and Sih (1966)

$$\begin{aligned} K_2 &= \frac{\tau\sqrt{(\pi b)}k^2k'}{B\{\sin^2\phi + (b/a)^2\cos^2\phi\}^{1/4}}\cos\phi \\ K_3 &= -\frac{\tau\sqrt{(\pi b)}(1-\nu)k^2}{B\{\sin^2\phi + (b/a)^2\cos^2\phi\}^{1/4}}\sin\phi \end{aligned} \tag{A3}$$

where a, b are the semi-major and minor axis of the ellipse and ϕ is the parametric angle describing the ellipse as

$$x = a\cos\phi;\quad z = b\sin\phi \tag{A4}$$

(crack lies on the plane $y = 0$) and

$$\begin{gathered} k' = b/a;\quad k^2 = 1-k'^2 \\ B = (k^2-\nu)E(k) + \nu k'^2K(k) \\ E(k) = \int_0^{\pi/2}\sqrt{(1-k^2\sin^2\phi)}\,d\phi;\quad K(k) = \int_0^{\pi/2}\frac{d\phi}{\sqrt{(1-k^2\sin^2\phi)}}. \end{gathered} \tag{A5}$$

Note that ϕ is not the real geometric polar angle of point (x, z) at the crack front, i.e. $\tan\phi \neq z/x$. This correction is needed both in drawings of Kassir and Sih (1966) and Tada *et al.* (1973).

It can be shown, to first-order accuracy in A/a_0 that in this case

$$a/a_0 = 1 + A/a_0;\quad b/a_0 = 1 - A/a_0;\quad k^2/B = \frac{4}{\pi(2-\nu)}\left(1 + \frac{4+\nu}{2(2-\nu)}A/a_0\right)$$

so that

$$\begin{aligned} K_2(\theta) &= \alpha\sqrt{(\pi a_0)}\cos\theta\left\{1 + \frac{A}{a_0}\left(3\cos^2\theta - 4 + \frac{1+\nu}{2-\nu}\right)\right\} \\ K_3(\theta) &= -(1-\nu)\alpha\sqrt{(\pi a_0)}\sin\theta\left\{1 + \frac{A}{a_0}\left(3\cos^2\theta + \frac{1+\nu}{2-\nu}\right)\right\}. \end{aligned} \tag{A6}$$

Substituting $n = 2$ into eqns (53), one may easily derive eqns (A6). Hence in both cases of $n = 1, 2$ the perturbation formulae match the analytical solutions.